Part 2: Question 1 (20 points)

Consider an economy with $I = 2$ consumers and $L = 2$ goods, where $X_i = \mathbb{R}_+^2$ for $i = 1, 2$. The preferences for each consumer can be represented by the following utility functions:

$$u_1(x_{11}, x_{21}) = x_{11} - \frac{1}{8}x_{21}^{-8}$$
$$u_2(x_{12}, x_{22}) = -\frac{1}{8}x_{12}^{-8} + x_{22}$$

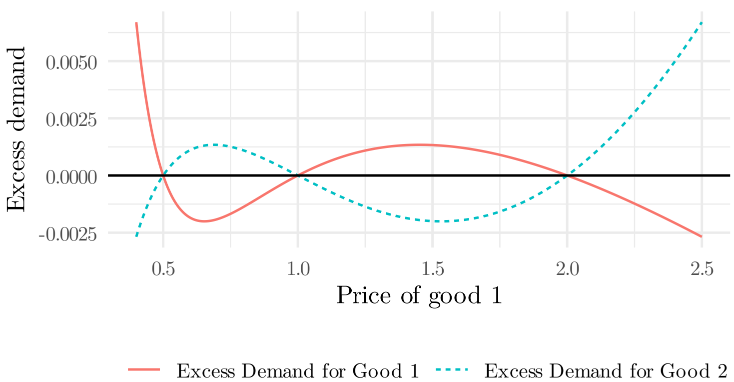
The initial endowment vectors are $\boldsymbol{\omega}_1 = \left(2, 2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right)$ and $\boldsymbol{\omega}_2 = \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}, 2\right)$. There is a single firm whose only production possibilities are free disposal: $Y_1 = -\mathbb{R}_+^2$. Each consumer owns a 50% stake in the firm. In this question, we normalize the price of good 2 to $p_2 = 1$.

Note: $2^{\frac{1}{9}} \approx 1.08006$ and $2^{\frac{8}{9}} \approx 1.85175$, so $2^{\frac{8}{9}} - 2^{\frac{1}{9}} \approx 0.77169 < \frac{7}{9} \approx 0.77778$.

(i) **[6 points]** Derive the aggregate demand function $\boldsymbol{z}(p_1)$ for this economy. Show that it is equal to:

$$z_1(p_1) = \left(\frac{1 - p_1}{p_1}\right)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{-\frac{8}{9}} + p_1^{-\frac{1}{9}}$$
$$z_2(p_1) = p_1^{\frac{1}{9}} + (p_1 - 1)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{\frac{8}{9}}$$

A graph of $z_1(p_1)$ and $z_2(p_1)$ for different values of p_1 is shown below:



(ii) **[4 points]** Show mathematically (i.e. not just referring to the graph) that $\boldsymbol{p}' = \left(\frac{1}{2}, 1\right)$, $\boldsymbol{p}''(1, 1)$ and $\boldsymbol{p}''' = (2, 1)$ are all equilibrium price vectors.

(iii) **[3 points]** Derive $\frac{\partial z_1(p_1)}{\partial p_1}$ and evaluate it at $p_1 = 1$. Show that it is positive.

(iv) **[2 points]** Does $\boldsymbol{z}(p_1)$ satisfy the gross-substitute property? Show whether it does or does not.

(v) [**2 points**] Is this economy regular? If so, what is its index? You may base your answer on the graph above.

(vi) [**3 points**] Suppose when determining the equilibrium prices of this economy that prices adjust over time according to: $\frac{dp_1(t)}{dt} = -z_1\left(p_1\left(t\right)\right)$. Which equilibria are locally stable? Is there system stability? You may base your answer on the graph above.

Solution:

(i) In any equilibrium, the firm will make zero profits so we may omit it from the consumers' budget constraints. Consumer 1's budget constraint, which will hold with equality, is:

$$p_1 x_{11} + x_{21} = 2p_1 + 2^{\frac{8}{9}} - 2^{\frac{1}{9}}$$

Solving for x_{11} and using it in the utility function yields the consumer's unconstrained problem:

$$\max_{x_{21}\geq 0} 2 + \frac{1}{p_1}\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - \frac{1}{p_1}x_{21} - \frac{1}{8}x_{21}^{-8}$$

The first-order condition (which will be interior) is:

$$x_{21}^{-9} = \frac{1}{p_1}$$

Solving for x_{21} gives $x_{21} = p_1^{\frac{1}{9}}$. Using this in the budget constraint to get the demand for good 1:

$$x_{11} = 2 + \frac{1}{p_1}\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - \frac{1}{p_1}p_1^{\frac{1}{9}} = 2 + \frac{1}{p_1}\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{-\frac{8}{9}}$$

Moving on to consumer 2, their budget is:

$$p_1 x_{12} + x_{22} = p_1\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) + 2$$

Their optimization problem is then:

$$\max_{x_{12}\geq 0} \; -\frac{1}{8}x_{12}^{-8} + p_1\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) + 2 - p_1 x_{12}$$

The first-order condition (which will be interior) is:

$$x_{12}^{-9} = p_1$$

Solving for x_{12} gives $x_{12} = p_1^{-\frac{1}{9}}$. Using this in the budget to get the demand for good 2:

$$x_{22} = p_1 \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) + 2 - p_1 x_{12} = p_1 \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) + 2 - p_1^{\frac{8}{9}}$$

So the excess demand for good 1 is then:

$$\begin{aligned} z_1(p_1) &= 2 + \frac{1}{p_1}\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{-\frac{8}{9}} + p_1^{-\frac{1}{9}} - 2 - \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) \\ &= \left(\frac{1-p_1}{p_1}\right)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{-\frac{8}{9}} + p_1^{-\frac{1}{9}} \end{aligned}$$

And for good 2:

$$\begin{aligned} z_2(p_1) &= p_1^{\frac{1}{9}} + p_1\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) + 2 - p_1^{\frac{8}{9}} - 2 - \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) \\ &= p_1^{\frac{1}{9}} + (p_1 - 1)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - p_1^{\frac{8}{9}} \end{aligned}$$

(ii) For an equilibrium, we need to find prices that make the excess demand function equal to zero for all goods. For good 1, we check that the candidate prices satisfy $z_1(p_1) = 0$.:

$$\begin{aligned} z_1\left(\frac{1}{2}\right) &= \left(\frac{1-\frac{1}{2}}{\frac{1}{2}}\right)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - \left(\frac{1}{2}\right)^{-\frac{8}{9}} + \left(\frac{1}{2}\right)^{-\frac{1}{9}} = \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - (2)^{\frac{8}{9}} + (2)^{\frac{1}{9}} = 0 \\ z_1(1) &= \left(\frac{1-1}{1}\right)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - (1)^{-\frac{8}{9}} + (1)^{-\frac{1}{9}} = 0 + \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - (1)^{\frac{8}{9}} + (1)^{\frac{1}{9}} = 0 \\ z_1(2) &= \left(\frac{1-2}{2}\right)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) - (2)^{-\frac{8}{9}} + (2)^{-\frac{1}{9}} = \underbrace{-\frac{1}{2}\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right)}_{=2^{-\frac{1}{9}} - 2^{-\frac{8}{9}}} - (2)^{-\frac{8}{9}} + (2)^{-\frac{1}{9}} = 0 \end{aligned}$$

We can also do this for good 2. Alternatively we can make use of the Lemma in class that stated that with Walras' law, once we have market clearing in all $L-1$ goods, we have market clearing in the last goods automatically.

(iii) The derivative of the aggregate demand of good 1 is:

$$\frac{\partial z_1(p_1)}{\partial p_1} = -\left(\frac{1}{p_1^2}\right)\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) + \frac{8}{9}p_1^{-\frac{17}{9}} - \frac{1}{9}p_1^{-\frac{10}{9}}$$

At $p_1 = 1$, this is:

$$-\left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) + \frac{8}{9} - \frac{1}{9} = \frac{7}{9} - \left(2^{\frac{8}{9}} - 2^{\frac{1}{9}}\right) > 0$$

The derivative of aggregate demand is positive at $p_1 = 1$.

(iv) We can't have a positive own-price derivative for any prices under gross substitutes (because of HD0 and GS implying negative off-diagonal terms in the Jacobian), so this aggregate demand function does not satisfy the gross substitute property. A more direct solution would be to find the sign on the cross-price derivative of the excess demand for good 2 at $p_1 = 1$:

$$\frac{\partial z_2(p_1)}{\partial p_1} = \frac{1}{9} p_1^{-\frac{8}{9}} + 2^{\frac{8}{9}} - 2^{\frac{1}{9}} - \frac{8}{9} p_1^{-\frac{1}{9}}$$

At $p = 1$, this is: $\frac{1}{9} + 2^{\frac{8}{9}} - 2^{\frac{1}{9}} - \frac{8}{9} = 2^{\frac{8}{9}} - 2^{\frac{1}{9}} - \frac{7}{9} < 0$. Because the cross-price derivative is negative, this aggregate excess demand function does not satisfy the gross substitute property.

(v) The economy is regular because the derivative of all excess demand functions is nonzero at each equilibrium. The indices of the equilibria are $+1$, -1 and $+1$, and the sum is 1.

(vi) Prices adjust in such a way that if excess demand is positive, the price of good 1 actually decreases (unlike the example in class). Therefore only the $p_1 = 1$ equilibrium is locally stable. If we start with $p_1 < \frac{1}{2}$, then prices converge to zero and we have no equilibrium. If we start with $p_1 > 2$, then prices go to infinity. If we start with $p_1 = \frac{1}{2}$ or $p_1 = 2$ it stays there, but any slight perturbation will lead prices to diverge from there. If we start with $p_1 \in \left(\frac{1}{2}, 2\right)$, then it converges to $p_1 = 1$.

Part 2: Question 2 (20 points)

Consider an economy with 2 consumers, 1 firm, and L goods. Each consumer $i = 1, 2$ has the consumption set $X_i = \mathbb{R}^L_+$ and has a preference relation $\succsim_i$ satisfying completeness, transitivity, convexity, continuity, and local nonsatiation. The single firm's production set Y_1 is convex and exhibits free disposal. The aggregate initial endowment vector satisfies $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$.

In this economy, there exists an allocation $(\boldsymbol{x}_1^\star, \boldsymbol{x}_2^\star, \boldsymbol{y}_1^\star)$ with $\boldsymbol{x}_i \gg \mathbf{0}$ for all i that is Pareto optimal.

(i) [**5 points**] Define $V_i = \{\boldsymbol{x}_i \in X_i : \boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star\}$. Show that V_i is a convex set. Do this by taking two arbitrary elements in V_i and showing that their convex combination is also contained in V_i.

Solution:

We need to show that if $\boldsymbol{x}_i \in V_i$ and $\boldsymbol{x}_i' \in V_i$, then $\boldsymbol{x}_i^\alpha = \alpha \boldsymbol{x}_i + (1-\alpha)\boldsymbol{x}_i' \in V_i$ for all $\alpha \in [0,1]$. First, by the convexity of X_i, $\boldsymbol{x}_i^\alpha \in X_i$. $\boldsymbol{x}_i, \boldsymbol{x}_i' \in V_i$ means $\boldsymbol{x}_i \succ_i \boldsymbol{x}_i^\star$ and $\boldsymbol{x}_i' \succ_i \boldsymbol{x}_i^\star$. Suppose wlog that $\boldsymbol{x}_i \succeq_i \boldsymbol{x}_i'$. Because preferences are convex: $\boldsymbol{x}_i^\alpha \succeq_i \boldsymbol{x}_i'$ $\forall \alpha \in [0,1]$ Then by transitivity $\boldsymbol{x}_i^\alpha \succ_i \boldsymbol{x}_i^\star$. Hence $\boldsymbol{x}_i^\alpha \in V_i$.

(ii) [**5 points**] Define the set:

$$V = \left\{\boldsymbol{x}_1 + \boldsymbol{x}_2 \in \mathbb{R}^L : \boldsymbol{x}_1 \in V_1 \text{ and } \boldsymbol{x}_2 \in V_2\right\}$$

This is the set of aggregate consumption bundles that can be split across the two consumers leaving them both strictly better off compared to the Pareto optimal allocation $(\boldsymbol{x}_1^\star, \boldsymbol{x}_2^\star, \boldsymbol{y}_1^\star)$.

Show that the set V is convex. Do this by taking two arbitrary elements in V and showing that their convex combination is also contained in V.

Solution:

Take $\boldsymbol{x}' = \boldsymbol{x}_1' + \boldsymbol{x}_2' \in V$ and and $\boldsymbol{x}'' = \boldsymbol{x}_1'' + \boldsymbol{x}_2'' \in V$. We want to show that $\forall \alpha \in [0,1]$ that $\alpha \boldsymbol{x}' + (1-\alpha)\boldsymbol{x}'' \in V$. $\boldsymbol{x}_1' \in V_1$ and $\boldsymbol{x}_1'' \in V_1$ and similarly for $\boldsymbol{x}_2'$ and $\boldsymbol{x}_2''$. Because V_1 and V_2 are convex, $\forall \alpha \in [0,1]$, $\boldsymbol{x}_1^\alpha = \alpha \boldsymbol{x}_1' + (1-\alpha)\boldsymbol{x}_1'' \in V_1$ and similarly $\boldsymbol{x}_2^\alpha \in V_2$. So, by the definition of V:

$$\begin{aligned}
\alpha \boldsymbol{x}' + (1-\alpha)\boldsymbol{x}'' &= \alpha\left(\boldsymbol{x}_1' + \boldsymbol{x}_2'\right) + (1-\alpha)\left(\boldsymbol{x}_1'' + \boldsymbol{x}_2''\right) \\
&= \alpha \boldsymbol{x}_1' + (1-\alpha)\boldsymbol{x}_1'' + \alpha \boldsymbol{x}_2' + (1-\alpha)\boldsymbol{x}_2'' \\
&= \boldsymbol{x}_1^\alpha + \boldsymbol{x}_2^\alpha
\end{aligned}$$

This is an element of V since it is the sum of two vectors which are each elements of V_1 and V_2.

(iii) [**5 points**] Define the set:

$$B = \left\{\boldsymbol{y}_1 + \bar{\boldsymbol{\omega}} \in \mathbb{R}^L : \boldsymbol{y}_1 \in Y_1\right\}$$

Show that B is convex. Do this by taking two arbitrary elements of B and showing their convex combination is also contained in B.

Solution:

Take $\boldsymbol{b}' = \boldsymbol{y}_1' + \bar{\boldsymbol{\omega}} \in B$ and $\boldsymbol{b}'' = \boldsymbol{y}_1'' + \bar{\boldsymbol{\omega}} \in B$. We want to show that $\boldsymbol{b}^\alpha = \alpha \boldsymbol{b}' +$

$(1-\alpha)\,\boldsymbol{b}'' \in B$ for any $\alpha \in [0,1]$. We first note that

$$\begin{aligned}\boldsymbol{b}^{\alpha} &= \alpha \boldsymbol{b}' + (1-\alpha)\,\boldsymbol{b}'' \\ &= \alpha \boldsymbol{y}_1' + \alpha \bar{\boldsymbol{\omega}} + (1-\alpha)\,\boldsymbol{y}_1'' + (1-\alpha)\,\bar{\boldsymbol{\omega}} \\ &= \underbrace{\alpha \boldsymbol{y}_1' + (1-\alpha)\,\boldsymbol{y}_1''}_{=\boldsymbol{y}_1^{\alpha}} + \bar{\boldsymbol{\omega}}\end{aligned}$$

Because $\boldsymbol{b}' \in B$, it follows that $\boldsymbol{y}_1' \in Y_1$. Similarly, because $\boldsymbol{b}'' \in B$, it follows that $\boldsymbol{y}_1'' \in Y_1$. By the convexity of Y_1, it follows that $\boldsymbol{y}_1^{\alpha} \in Y_1$. Therefore $\boldsymbol{b}^{\alpha} = \boldsymbol{y}_1^{\alpha} + \bar{\boldsymbol{\omega}} \in B$.

(iv) [**5 points**] Show that the sets V and B are disjoint. *Hint:* Use proof by contradiction.
Solution:
To show that V and B are disjoint, suppose they were not. Then there is a non-empty intersection: there is an aggregate bundle $\boldsymbol{x}'$ in both V and B. V is the set of bundles that can make both consumers better off compared to the Pareto optimal allocation $(\boldsymbol{x}_1^{\star}, \boldsymbol{x}_2^{\star}, \boldsymbol{y}_1^{\star})$. B is the set of feasible bundles. If $\boldsymbol{x}'$ is in both V and B, then it is both feasible to produce and can be split in a way to make both consumers better off compared to the Pareto optimal allocation $(\boldsymbol{x}_1^{\star}, \boldsymbol{x}_2^{\star}, \boldsymbol{y}_1^{\star})$. But this contradicts that $(\boldsymbol{x}_1^{\star}, \boldsymbol{x}_2^{\star}, \boldsymbol{y}_1^{\star})$ is Pareto optimal in the first place, as we have found another allocation this is both feasible and strictly preferred.